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A groupoid approach to spaces of generalized connections

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Abstract

The quantum completion $\bar{\mathcal{A}}$ of the space of connections in a manifold can be seen as the set of all morphisms from the groupoid of the edges of the manifold to the (compact) gauge group. This algebraic construction generalizes an analogous description of the gauge-invariant quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ of Ashtekar and Isham, clarifying the relation between the two spaces. We present a description of the groupoid approach which brings the gauge-invariant degrees of freedom to the foreground, thus making the action of the gauge group more transparent. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Theories of connections play an important role in the description of fundamental interactions, including Yang–Mills theories [21], Chern-Simons theories [22] and gravity in the Ashtekar formulation [1]. Typically in such cases, the classical configuration space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations is an infinite-dimensional non-linear space of great complexity, challenging the usual field quantization techniques.

Having in mind a rigorous quantization of theories of connections and eventually of gravity, methods of functional calculus in an extension of $\mathcal{A}/\mathcal{G}$ were developed over the last decade. For a compact gauge group G, the extension $\overline{\mathcal{A}/\mathcal{G}}$ introduced by Ashtekar and Isham [2] is a natural compact measurable space, allowing the construction of well-defined diffeomorphism invariant measures [3,4,8]. Like in the case of measures in infinite-dimensional

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linear spaces, which appear in the context of constructive quantum scalar field theory, interesting measures in $\overline{\mathcal{A}/\mathcal{G}}$ are expected to be supported not on classical configurations but on genuine (distributional-like) generalized connections (this was indeed proven to be the case for the Ashtekar–Lewandowski measure [3], in [17,18]).

In later developments, Baez [7] considered an extension $\bar{\mathcal{A}}$ of the space $\mathcal{A}$ of smooth connections. In this case one still has to divide by the appropriate action of gauge transformations. Besides being equally relevant for integral calculus, the space $\bar{\mathcal{A}}$ is particularly useful for the definition of differential calculus in $\overline{\mathcal{A}/\mathcal{G}}$, fundamental in the construction of quantum observables [5].

The construction of both $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$ rely crucially on the use of Wilson variables (and generalizations), bringing to the foreground the important role of parallel transport defined by certain types of curves. In this work we will consider only the case of piecewise analytic curves, for which the formalism was originally introduced, although most of the arguments apply equally well to the more general piecewise smooth case developed by Baez and Sawin [10] and later by Lewandowski and Thiemann [16] (see also [19] and [11,12] for more recent developments). For both $\bar{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ one considers functions on $\mathcal{A}$ of the form

$$\mathcal{A} \ni A \mapsto F(h(c_1, A), \ldots, h(c_n, A)), \tag{1}$$

where $h(c, A)$ denotes the parallel transport defined by the curve c and $F : G^n \to \mathbb{C}$ is a continuous function. In the case of $\overline{\mathcal{A}/\mathcal{G}}$ only closed curves — loops — are needed, producing gauge-invariant functions, or functions on $\mathcal{A}/\mathcal{G}$. These functions are sufficient to define (overcomplete) coordinates on $\mathcal{A}/\mathcal{G}$ [2]. For compact G, the set of all functions (1) is naturally a normed commutative $*$-algebra with identity. The completion of such an algebra is, therefore, a commutative unital C^*-algebra and, according to Gelfand theory, this C^*-algebra can be seen as the algebra of continuous functions on a compact space called the spectrum of the algebra. The spectrum of the above algebras — $\overline{\mathcal{A}/\mathcal{G}}$ for the closed curves case and $\bar{\mathcal{A}}$ for the general open curves case — are natural completions of $\mathcal{A}/\mathcal{G}$ and $\mathcal{A}$, respectively, and appear as good candidates to replace them in the quantum context.

To a large extent, the definition of functional calculus on $\overline{\mathcal{A}/\mathcal{G}}$ rely on the fact that, while being extremely complex spaces, both $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$ can be seen as projective limits of families of finite-dimensional compact manifolds [3,4,17] (see also [7,10,16] for a formulation in terms of inductive limits). This projective characterization gives us a great deal of control over the spaces $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$, allowing the construction of measures and vector fields starting from corresponding structures on the compact finite-dimensional spaces in the projective families [3–5,7,17].

The projective approach leads also to an interesting interpretation of generalized connections. For the case of $\overline{\mathcal{A}/\mathcal{G}}$, a distinguished group of equivalence classes of loops, called the hoop group $\mathcal{HG}$ [3], plays an important role, in the sense that $\overline{\mathcal{A}/\mathcal{G}}$ can be identified with the space $\mathrm{Hom}[\mathcal{HG}, G]/G$ of all homomorphisms (modulo conjugation) from $\mathcal{HG}$ to G, with the topology on $\mathrm{Hom}[\mathcal{HG}, G]/G$ being induced by a projective family labeled by finitely generated subgroups of $\mathcal{HG}$. As pointed out by Baez [9], for $\bar{\mathcal{A}}$ a similar role is played by a certain groupoid. In our opinion, however, this groupoid associated to open curves has not yet occupied the place it deserves in the literature, possibly due to the fact that groupoids have been introduced in the current mathematical physics literature only recently. Recall that a groupoid is a category such that all arrows are invertible. Therefore, a groupoid generalizes the notion of a group, in the sense that a binary operation with inverse is defined, the difference being that not all pairs of elements can be composed.

In Section 2 of this work, we consider the projective characterization of $\bar{\mathcal{A}}$ using the language of groupoids from the very beginning. This amounts to putting the usual approach using graphs [5] in an appropriate algebraic framework, in a natural generalization of the hoop group approach. Using this formalism, we show in Section 3 that the quotient of $\bar{\mathcal{A}}$ by the action of the gauge group is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$. This new proof, establishing directly the equivalence at the projective limit level, seems to us more transparent than the proof one can obtain by combining results from [3–5,7,17].

2. Groupoid-projective formulation of $\bar{\mathcal{A}}$

2.1. Edge groupoid

Let Σ be an analytic, connected and orientable d-manifold. Let us consider the set $\mathcal{E}$ of all continuous, oriented and piecewise analytic parametrized curves in Σ, i.e. maps

$$c : [0, t_1] \cup \cdots \cup [t_{n-1}, 1] \to \Sigma,$$

which are continuous in all the domain $[0, 1]$, analytic in the closed intervals $[t_k, t_{k+1}]$ and such that the images $c(]t_k, t_{k+1}[)$ of the open intervals $]t_k, t_{k+1}[$ are submanifolds embedded in Σ. In the set $\mathcal{E}$ of all such curves one may define the following maps. Let $\sigma : \mathcal{E} \to \Sigma$ be the map given by $\sigma(c) = c([0, 1])$, $c \in \mathcal{E}$. The maps s (source) and r (range) are defined, respectively, by $s(c) = c(0)$, $r(c) = c(1)$. Given two curves $c_1, c_2 \in \mathcal{E}$ such that $s(c_2) = r(c_1)$, let $c_2c_1 \in \mathcal{E}$ denote the natural composition given by

$$(c_2c_1)(t) = \begin{cases} c_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ c_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

This composition defines a binary operation in a well-defined subset of $\mathcal{E} \times \mathcal{E}$. Consider also the operation $c \mapsto c^{-1}$ given by $c^{-1}(t) = c(1-t)$. Strictly speaking, the composition of parametrized curves is not associative, since the curves $(c_3c_2)c_1$ and $c_3(c_2c_1)$ are related by a reparametrization, i.e. by an orientation preserving piecewise analytic diffeomorphism $[0, 1] \to [0, 1]$. Similarly, the curve c^{-1} is not the inverse of the curve c. Following [2,3,9], we describe next an appropriate equivalence relation in $\mathcal{E}$. The corresponding set of equivalence classes is a well-defined groupoid [9], generalizing the group of hoops introduced by Ashtekar and Lewandowski [3].

Let G be a (finite-dimensional) connected and compact Lie group and let $P(\Sigma, G)$ be a principal G-bundle over Σ. For simplicity, we assume that the bundle is trivial and that a fixed trivialization has been chosen. Let $\mathcal{A}$ be the space of smooth connections on this bundle. The parallel transport associated with a given connection $A \in \mathcal{A}$ and a given curve $c \in \mathcal{E}$ will be denoted by $h(c, A)$.

Definition 1. Two elements c and c' of $\mathcal{E}$ are said to be equivalent if

1. $s(c) = s(c'), r(c) = r(c')$;
2. $h(c, A) = h(c', A) \quad \forall A \in \mathcal{A}$.

It is obvious that two curves related by a reparametrization are equivalent. Two curves c and c' which can be written in the form $c = c_2c_1$, $c' = c_2c_3^{-1}c_3c_1$ are also equivalent. It can be shown that, for compact non-commutative Lie groups G, these two conditions are equivalent to (2) (see, e.g. [4,16]). Thus, in the context of non-commutative compact Lie groups, the equivalence relation above is independent of the group.

We will consider non-commutative groups from now on and denote the set of all above defined equivalence classes by $\mathcal{EG}$. It is clear by (1) that the maps s and r are well-defined in $\mathcal{EG}$. The map σ can still be defined for special elements called edges. By edges we mean elements $e \in \mathcal{EG}$ which are equivalence classes of analytic (in all domain) curves $c : [0, 1] \to \Sigma$. It is clear that the images $c_1([0, 1])$ and $c_2([0, 1])$ of two equivalent analytic curves coincide and, therefore, we define $\sigma(e)$ as being $\sigma(c)$, where c is any analytic curve in the class of the edge e.

We discuss next the natural groupoid structure on the set $\mathcal{EG}$. We will follow the terminology of category theory and refer to elements of $\mathcal{EG}$ as arrows.

The composition of arrows is defined by the composition of elements of $\mathcal{E}$: if $\gamma, \gamma' \in \mathcal{EG}$ are such that $r(\gamma) = s(\gamma')$ one defines $\gamma'\gamma$ as the equivalence class of $c'c$, where c (resp. c') belongs to the class γ (γ'). The independence of this composition with respect to the choice of representatives follows from $h(c'c, A) = h(c', A)h(c, A)$ and from condition (2) above. The composition in $\mathcal{EG}$ is now associative, since $(c_3c_2)c_1$ and $c_3(c_2c_1)$ belong to the same equivalence class.

The points of Σ are called objects in this context. Objects are in one-to-one correspondence with identity arrows: given $x \in \Sigma$ the corresponding identity $\mathbf{1}_x \in \mathcal{EG}$ is the equivalence class of $c^{-1}c$, with $c \in \mathcal{E}$ such that $s(c) = x$. If γ is the class of c then γ^{-1} is the class of c^{-1}. It is clear that $\gamma^{-1}\gamma = \mathbf{1}_{s(\gamma)}$ and $\gamma\gamma^{-1} = \mathbf{1}_{r(\gamma)}$.

One, therefore, has a well-defined groupoid, whose set of objects is Σ and whose set of arrows is $\mathcal{EG}$. As usual, we will use the same notation — $\mathcal{EG}$ — both for the set of arrows and for the groupoid. Notice that every element $\gamma \in \mathcal{EG}$ can be obtained as a composition of edges. Therefore, the groupoid $\mathcal{EG}$ is generated by the set of edges, although it is not freely generated, since composition of edges may produce new edges.

For $x, y \in \Sigma$, let

$$\text{Hom}[x, y] := \{\gamma \in \mathcal{EG} | s(\gamma) = x, r(\gamma) = y\} \tag{2}$$

be the set of all arrows starting at x and ending at y. It is clear that $\text{Hom}[x, x]$ is a group $\forall x \in \Sigma$. Since the manifold Σ is taken to be connected, the groupoid $\mathcal{EG}$ is also connected, i.e. $\text{Hom}[x, y]$ is a non-empty set, for every pair $x, y \in \Sigma$. In this case, any two groups $\text{Hom}[x, x]$ and $\text{Hom}[y, y]$ are isomorphic. Let us fix a point $x_0 \in \Sigma$ and consider the group $\text{Hom}[x_0, x_0]$. This group is precisely the so-called hoop group $\mathcal{HG}$ [3], whose elements are equivalence classes of piecewise analytic loops. The elements of $\text{Hom}[x_0, x_0]$ are called hoops and the identity arrow $\mathbf{1}_{x_0}$ will be called the trivial hoop.

Given that $\mathcal{EG}$ is connected, its elements may be written as compositions of elements of $\mathrm{Hom}[x_0, x_0]$ and of an appropriate subset of the set of all arrows.

Lemma 1. *Suppose that an unique arrow $\gamma_x \in \mathrm{Hom}[x_0, x]$ is given for each $x \in \Sigma$, γ_{x_0} being the trivial hoop. Then for every $\gamma \in \mathcal{EG}$ there is a uniquely defined $\beta \in \mathrm{Hom}[x_0, x_0]$ such that*

$$\gamma = \gamma_{r(\gamma)}\beta\gamma_{s(\gamma)}^{-1}. \tag{3}$$

This result can be obviously adapted for any connected subgroupoid $\Gamma \subset \mathcal{EG}$. The converse of this result is the following lemma, where $\mathrm{Hom}_\Gamma[x_0, x_0]$ denotes the subgroup of the hoops that belong to Γ.

Lemma 2. *Let F be a subgroup of $\mathrm{Hom}[x_0, x_0]$ and $X \subset \Sigma$ be a subset of Σ such that $x_0 \in X$. Suppose that an unique arrow $\gamma_x \in \mathrm{Hom}[x_0, x]$ is given for each $x \in X$, γ_{x_0} being the trivial hoop. Then the set Γ of all arrows of the form $\gamma_x\beta\gamma_y^{-1}$, with $\beta \in F$ and $x, y \in X$, is a connected subgroupoid of $\mathcal{EG}$, and the group $\mathrm{Hom}_\Gamma[x_0, x_0]$ coincides with F.*

Proof. To prove that Γ is subgroupoid it is sufficient to show that (i) every arrow $\gamma \in \Gamma$ is invertible in Γ and (ii) that the composition $\gamma\gamma'$ belongs to Γ, for every $\gamma, \gamma' \in \Gamma$ such that $\gamma\gamma'$ is defined on $\mathcal{EG}$. The inverse of $\gamma_x\beta\gamma_y^{-1}$ is $\gamma_y\beta^{-1}\gamma_x^{-1} \in \Gamma$, proving (i). As for (ii), notice that given $\gamma = \gamma_x\beta\gamma_y^{-1}$ and $\gamma' = \gamma_{x'}\beta'\gamma_{y'}^{-1}$, the composition $\gamma\gamma'$ is defined if and only if $y = x'$, and therefore $\gamma\gamma' = \gamma_x(\beta\beta')\gamma_{y'}^{-1}$ belongs to Γ, since F is a group. The groupoid Γ is connected, given that every object $x \in X$ is connected to x_0 by an arrow. If $\gamma = \gamma_x\beta\gamma_y^{-1}$ belongs to $\mathrm{Hom}_\Gamma[x_0, x_0]$ then $x = y = x_0$ and $\gamma = \beta \in F$. Conversely, it is obvious that $F \subset \mathrm{Hom}_\Gamma[x_0, x_0]$. □

2.2. $\bar{\mathcal{A}}$ as a projective limit

By the very definition of $\mathcal{EG}$ (see condition (2) in Definition 1), the parallel transport is well-defined for any element of $\mathcal{EG}$. To emphasize the algebraic role of connections and to simplify the notation, we will denote by $A(\gamma)$ the parallel transport $h(c, A)$ defined by $A \in \mathcal{A}$ and any curve c in the equivalence class $\gamma \in \mathcal{EG}$. Let us recall that the bundle $P(\Sigma, G)$ is assumed to be trivial, and therefore $A(\gamma) \equiv h(c, A)$ defines an element of the group G. For every connection $A \in \mathcal{A}$, the map from $\mathcal{EG}$ to G given by

$$\gamma \mapsto A(\gamma) \tag{4}$$

is a groupoid morphism, i.e., $A(\gamma'\gamma) = A(\gamma')A(\gamma)$ and $A(\gamma^{-1}) = A(\gamma)^{-1}$. Thus, there is a well-defined injective but not surjective [2,3,7,15] map from $\mathcal{A}$ to the set $\mathrm{Hom}[\mathcal{EG}, G]$ of all morphisms from $\mathcal{EG}$ to G, through which $\mathcal{A}$ can be seen as a proper subset of $\mathrm{Hom}[\mathcal{EG}, G]$. It turns out that $\mathrm{Hom}[\mathcal{EG}, G]$, when equipped with an appropriate topology, is homeomorphic to the space $\bar{\mathcal{A}}$ of generalized connections [4,9,17]. This identification can be proved using the fact that $\mathrm{Hom}[\mathcal{EG}, G]$ is the projective limit of a projective family labeled by graphs in the manifold Σ [5,6]. In what follows we will rephrase the projective characterization of Hom[$\mathcal{EG}$, G] using the language of groupoids. We start with the set of labels for the projective family leading to Hom[$\mathcal{EG}$, G], using the notion of independent edges [3].

Definition 2. A finite set $\{e_1, \ldots, e_n\}$ of edges is said to be independent if the edges e_i can intersect each other only at the points $s(e_i)$ or $r(e_i)$, $i = 1, \ldots, n$.

The edges in an independent set are, in particular, algebraically independent, i.e. it is not possible to produce identity arrows by (non-trivial) compositions of the edges and their inverses. Our condition of independent sets is of course stronger than the condition of algebraic independence.

Let us denote by $\mathcal{EG}\{e_1, \ldots, e_n\}$ the subgroupoid of $\mathcal{EG}$ generated by the independent set $\{e_1, \ldots, e_n\}$, i.e. $\mathcal{EG}\{e_1, \ldots, e_n\}$ is the smallest subgroupoid containing all the edges e_i, or explicitly, the subgroupoid whose objects are all the points $s(e_i)$ and $r(e_i)$ and whose arrows are all possible compositions of edges e_i and their inverses. Groupoids of this type are freely generated, given the algebraic independence of the edges.

In what follows we will denote by $\mathcal{L}$ the set of all subgroupoids for which there exists a finite set of independent edges such that $L = \mathcal{EG}\{e_1, \ldots, e_n\}$. Clearly, the sets $\{e_1, \ldots, e_n\}$ and $\{e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}\}$, where $\epsilon_j = \pm 1$ (i.e. $e_j^{\epsilon_j} = e_j$ or e_j^{-1}) generate the same subgroupoid, and this is the only ambiguity in the choice of the set of generators of a given groupoid $L \in \mathcal{L}$. Thus, a groupoid $L \in \mathcal{L}$ is uniquely defined by a set $\{\sigma(e_1), \ldots, \sigma(e_n)\}$ of images of a set of independent edges. Notice that the union of the images $\sigma(e_i)$ is a graph in the manifold Σ, thus establishing the relation with the approach used in [5,7,8].

Let us consider in the set $\mathcal{L}$ the partial-order relation defined by inclusion, i.e. given $L, L' \in \mathcal{L}$, we will say that $L' \geq L$ if and only if L is a subgroupoid of L'. Recall that L is said to be a subgroupoid of L' if and only if all objects of L are objects of L' and for any pair of objects x, y of L every arrow from x to y is an arrow of L'. It is easy to see that $\mathcal{L}$ is a directed set with respect to the latter partial-order, meaning that for any given L and L' in $\mathcal{L}$ there exists $L'' \in \mathcal{L}$ such that $L'' \geq L$ and $L'' \geq L'$. We will not repeat here the arguments leading to this conclusion; the crucial fact is that for every finitely generated subgroupoid $\Gamma \subset \mathcal{EG}$ there is an element $L \in \mathcal{L}$ such that Γ is a subgroupoid of L, which can be easily proved in the piecewise analytic case [3].

Let us now consider the projective family. For each $L \in \mathcal{L}$, let $\mathcal{A}_L :=$ Hom[L, G] be the set of all morphisms from the groupoid L to the group G. We will show next that the family of spaces $\mathcal{A}_L$, $L \in \mathcal{L}$, is a so-called compact Hausdorff projective family (see [4]), meaning that each of the spaces $\mathcal{A}_L$ is a compact Hausdorff space and that given $L, L' \in \mathcal{L}$ such that $L' \geq L$ there exists a surjective and continuous projection $p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$ such that

$$p_{L,L''} = p_{L,L'} \circ p_{L',L''} \quad \forall L'' \geq L' \geq L. \tag{5}$$

There is a well-defined notion of limit of the family of spaces $\mathcal{A}_L$ — the projective limit — which is also a compact Hausdorff space.

Given $L \in \mathcal{L}$, let $\{e_1, \ldots, e_n\}$ be a set of independent edges that freely generates the groupoid L. Since the morphisms $L \to G$ are uniquely determined by the images of the generators of L, one gets a bijection $\rho_{e_1,\ldots,e_n} : \mathcal{A}_L \to G^n$, given by

$$\mathcal{A}_L \ni \bar{A} \mapsto (\bar{A}(e_1), \ldots, \bar{A}(e_n)) \in G^n. \tag{6}$$

Through this identification with G^n, the space $\mathcal{A}_L$ acquires a topology with respect to which it is a compact Hausdorff space. Notice that the topology induced in $\mathcal{A}_L$ is independent of the choice of the generators (including ordering), since maps of the form

$$(g_1, \ldots, g_n) \mapsto (g_{k_1}^{\epsilon_{k_1}}, \ldots, g_{k_n}^{\epsilon_{k_n}}), \tag{7}$$

where $(k_1, \ldots, k_n)$ is a permutation of $(1, \ldots, n)$ and $\epsilon_{k_i} = \pm 1$, are homeomorphisms $G^n \to G^n$. For $L' \geq L$ let us define the projection $p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$ as the map that sends each element of $\mathcal{A}_{L'}$ to its restriction to L. It is clear that (5) is satisfied. We will now show that the maps $p_{L,L'}$ are surjective and continuous. Let $\{e_1, \ldots, e_n\}$ be generators of L and $\{e'_1, \ldots, e'_m\}$ be generators of $L' \geq L$. Let us consider the decomposition of the edges e_i in terms of the edges e'_j:

$$e_i = \prod_j (e'_{r_{ij}})^{\epsilon_{ij}}, \quad i = 1, \ldots, n, \tag{8}$$

where r_{ij} and ϵ_{ij} take values in the sets $\{1, \ldots, m\}$ and $\{1, -1\}$, respectively. An arbitrary element of $\mathcal{A}_L$ is identified by the images $(h_1, \ldots, h_n) \in G^n$ of the ordered set of generators $(e_1, \ldots, e_n)$. The map $p_{L,L'}$ will be surjective if and only if there are $(g_1, \ldots, g_m) \in G^m$ such that

$$h_i = \prod_j g_{r_{ij}}^{\epsilon_{ij}} \quad \forall i. \tag{9}$$

These conditions can indeed be satisfied, since they are independent. In fact, since the edges $\{e_1, \ldots, e_n\}$ are independent, a given edge e'_k can appear at most once (in the form e'_k or e'^{-1}_k) in the decomposition (8) of a given e_i. To prove continuity notice that, through the identification (6), the map $p_{L,L'}$ corresponds to the projection $\pi_{n,m} : G^m \to G^n$:

$$G^m \ni (g_1, \ldots, g_m) \overset{\pi_{n,m}}{\mapsto} \left(\prod_j g_{r_{1j}}^{\epsilon_{1j}}, \ldots, \prod_j g_{r_{nj}}^{\epsilon_{nj}} \right) \in G^n, \tag{10}$$

which is continuous.

The projective limit of the family $\{\mathcal{A}_L, p_{L,L'}\}_{L,L' \in \mathcal{L}}$ is the subset $\mathcal{A}_\infty$ of the Cartesian product $\mathsf{X}_{L \in \mathcal{L}} \mathcal{A}_L$ of those elements $(A_L)_{L \in \mathcal{L}}$ satisfying the following consistency conditions:

$$p_{L,L'} A_{L'} = A_L \quad \forall L' \geq L. \tag{11}$$

The Cartesian product is a compact Hausdorff space with the Tychonov product topology. Given the continuity of the projections $p_{L,L'}$, the projective limit $\mathcal{A}_\infty$ is a closed subset [4,17] and, therefore, is also a compact Hausdorff space. Explicitly, the induced topology in $\mathcal{A}_\infty$ is the weakest topology such that all the following projections are continuous:

$$p_L : \quad \mathcal{A}_\infty \to \mathcal{A}_L, \qquad (A_L)_{L \in \mathcal{L}} \mapsto A_L. \tag{12}$$

The proof that the projective limit $\mathcal{A}_\infty$ coincides with the set of all groupoid morphisms Hom$[\mathcal{EG}, G]$ follows essentially the same steps as the proof of the well-known fact that the algebraic dual of any vector space is a projective limit, and therefore will not be presented here (see, e.g. [3,4,17] for the closely related case of the space of generalized connections modulo gauge transformations). It is interesting to note that $\mathrm{Hom}[\mathcal{E}\mathcal{G}, G]$ can be seen as being dual (in a non-linear sense) to the groupoid $\mathcal{E}\mathcal{G}$. In what follows we will identify $\mathcal{A}_\infty$ with $\mathrm{Hom}[\mathcal{E}\mathcal{G}, G]$. For simplicity, we will refer to the induced topology on $\mathrm{Hom}[\mathcal{E}\mathcal{G}, G]$ as the Tychonov topology.

3. Relation between $\bar{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ in the groupoid-projective approach

In this section, we will study the relation between the space of generalized connections considered above and the space $\overline{\mathcal{A}/\mathcal{G}}$ of generalized connections modulo gauge transformations [3,4], from the point of view of projective techniques. The gauge transformations act naturally in $\mathrm{Hom}[\mathcal{E}\mathcal{G}, G]$ and, as expected, the quotient of $\mathrm{Hom}[\mathcal{E}\mathcal{G}, G]$ by this action is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$. The proof presented here complements the results in [3–5,7,17] and clarifies the relation between the two spaces. The introduction of the groupoid $\mathcal{E}\mathcal{G}$ plays a relevant simplifying role in this result.

3.1. Gauge transformations, $\bar{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$

We start with a brief review of the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$ [3,4,17]. A finite set of hoops $\{\beta_1, \ldots, \beta_n\}$ is said to be independent if each hoop β_i contains an edge which is traversed only once and which is shared by any other hoop at most at a finite number of points. In the hoop formulation the projective family is labeled by certain "tame" subgroups of the hoop group $\mathcal{H}\mathcal{G} \equiv \mathrm{Hom}[x_0, x_0]$, which are subgroups freely generated by finite sets of independent hoops. We will denote the family of such subgroups by $\mathcal{S}_{\mathcal{H}}$. For each $S \in \mathcal{S}_{\mathcal{H}}$ one considers the set χ_S of all homomorphisms $S \to G$

$$\chi_S := \mathrm{Hom}[S, G]. \tag{13}$$

The sets χ_S can be identified with powers of G and the family $\{\chi_S\}_{S\in\mathcal{S}_{\mathcal{H}}}$ is a compact Hausdorff projective family, whose projective limit is $\mathrm{Hom}[\mathcal{H}\mathcal{G}, G]$, the set of all homomorphisms from the $\mathcal{H}\mathcal{G}$ to G [4]. By means of the projective family, the space $\mathrm{Hom}[\mathcal{H}\mathcal{G}, G]$ is equipped with a Tychonov-like topology, namely the weakest topology such that all the natural projections

$$p_S : \mathrm{Hom}[\mathcal{H}\mathcal{G}, G] \to \chi_S, \quad S \in \mathcal{S}_{\mathcal{H}}, \tag{14}$$

defined by restriction to $S \subset \mathcal{H}\mathcal{G}$, are continuous.

The group G acts continuously on $\mathrm{Hom}[\mathcal{H}\mathcal{G}, G]$ in the following way [4]:

$$\mathrm{Hom}[\mathcal{H}\mathcal{G}, G] \times G \ni (H, g) \mapsto H_g : H_g(\beta) = g^{-1}H(\beta)g \quad \forall \beta \in \mathcal{H}\mathcal{G}. \tag{15}$$

This action corresponds to the non-trivial part of the action of the group of generalized local gauge transformations (see below). It is a well-established fact that the quotient space $\mathrm{Hom}[\mathcal{H}\mathcal{G}, G]/G$ is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$, the "quantum configuration space" which replaces the classical configuration space $\mathcal{A}/\mathcal{G}$ in the Isham–Ashtekar–Lewandowski approach to the quantization of theories of connections [2–6,17].

Let us consider now the corresponding action of local gauge transformations on generalized connections. The group of local gauge transformations associated with the structure group G is the group $\mathcal{G}$ of all smooth maps $g : \Sigma \to G$, acting on smooth connections as follows:

$$\mathcal{A} \ni A \mapsto g^{-1}Ag + g^{-1}\,\mathrm{d}g,$$

where d denotes the exterior derivative. The corresponding action on parallel transports $A(\gamma)$ defined by $A \in \mathcal{A}$ and $\gamma \in \mathcal{EG}$ is given by

$$A(\gamma) \mapsto g(x_2)^{-1}A(\gamma)g(x_1), \quad g \in \mathcal{G}, \tag{16}$$

where $x_1 = s(\gamma)$, $x_2 = r(\gamma)$. Let us consider the extension $\bar{\mathcal{G}}$ of $\mathcal{G}$,

$$\bar{\mathcal{G}} = \mathrm{Map}[\Sigma, G] = G^{\Sigma} \cong \mathsf{X}_{x\in\Sigma} G_x \tag{17}$$

of all maps $g : \Sigma \to G$, not necessarily smooth or even continuous. This group $\bar{\mathcal{G}}$ of “generalized local gauge transformations” acts naturally on the space of generalized connections $\mathrm{Hom}[\mathcal{EG}, G]$,

$$\mathrm{Hom}[\mathcal{EG}, G] \times \bar{\mathcal{G}} \ni (\bar{A}, g) \mapsto \bar{A}_g \in \mathrm{Hom}[\mathcal{EG}, G], \tag{18}$$

where

$$\bar{A}_g(\gamma) = g(r(\gamma))^{-1}\bar{A}(\gamma)g(s(\gamma)) \quad \forall\gamma \in \mathcal{EG}, \tag{19}$$

generalizing (16). It is natural to consider the quotient of $\mathrm{Hom}[\mathcal{EG}, G]$ by the action of $\bar{\mathcal{G}}$, since $\mathrm{Hom}[\mathcal{EG}, G]$ is also made of all the morphisms $\mathcal{EG} \to G$, without any continuity condition. The group $\bar{\mathcal{G}}$ is compact Hausdorff (with the product topology) and its action is continuous [4,5]. Therefore, $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is also a compact Hausdorff space.

Let us consider the compact space $\bar{\mathcal{A}}$ as introduced by Baez [7], e.g. as the Gelfand spectrum of a commutative unital C^*-algebra. According to Gelfand theory, the original C^*-algebra can be identified with the algebra $C(\bar{\mathcal{A}})$ of continuous functions in $\bar{\mathcal{A}}$. The group of local gauge transformations acts on $C(\bar{\mathcal{A}})$ and the subspace $C^{\mathcal{G}}(\bar{\mathcal{A}}) \subset C(\bar{\mathcal{A}})$ of gauge-invariant functions is also a unital commutative C^*-algebra, whose spectrum we will denote by $\bar{\mathcal{A}}/\bar{\mathcal{G}}$.

One, therefore, has four extensions of the classical configuration space $\mathcal{A}/\mathcal{G}$, namely $\overline{\mathcal{A}/\mathcal{G}}$, $\bar{\mathcal{A}}/\bar{\mathcal{G}}$, $\mathrm{Hom}[\mathcal{HG}, G]/G$ and $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$. The first two spaces are tied to the C^*-algebra formalism, whereas the last two appear in the context of projective methods. As expected, all these spaces are naturally homeomorphic. Let us consider the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{A}/\mathcal{G}} & \leftrightarrow & \mathrm{Hom}[\mathcal{HG}, G]/G \\ \updownarrow & & \\ \bar{\mathcal{A}}/\bar{\mathcal{G}} & \leftrightarrow & \mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}. \end{array}$$

The correspondence between $\overline{\mathcal{A}/\mathcal{G}}$ and $\mathrm{Hom}[\mathcal{HG}, G]/G$ was established in [17]. The generalization of this result given in [4] produces a homeomorphism between $\bar{\mathcal{A}}$ and $\mathrm{Hom}[\mathcal{EG}, G]$. It is not difficult to show that this homeomorphism is equivariant, leading to a homeomorphism between $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ and $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$, as suggested in [5]. The correspondence between $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ follows from results in [7].

In the next section, we will show directly (i.e. without using the diagram above) that $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is homeomorphic to $\mathrm{Hom}[\mathcal{HG}, G]/G$. The relevance of this new proof of a known result lies in the clear relation established between $\mathrm{Hom}[\mathcal{EG}, G](\cong \bar{\mathcal{A}})$ and $\mathrm{Hom}[\mathcal{HG}, G]/G(\cong \overline{\mathcal{A}/\mathcal{G}})$, without having to rely on the characterization of these spaces as spectra of C^*-algebras.

3.2. *Equivalence of the projective characterizations of $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$*

Since $\mathcal{HG} \equiv \mathrm{Hom}[x_0, x_0]$ is a subgroup of the groupoid $\mathcal{EG}$, a projection $\mathcal{P}$: $\mathrm{Hom}[\mathcal{EG}, G] \to \mathrm{Hom}[\mathcal{HG}, G]$, given by the restriction of elements of $\mathrm{Hom}[\mathcal{EG}, G]$ to the group $\mathcal{HG}$, is naturally defined. We will show that this projection is surjective and equivariant with respect to the actions of $\bar{\mathcal{G}}$ on $\mathrm{Hom}[\mathcal{EG}, G]$ and $\mathrm{Hom}[\mathcal{HG}, G]$, thus defining a map $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}} \to \mathrm{Hom}[\mathcal{HG}, G]/G$ which is in fact a bijection. We will also show that the latter map and its inverse are continuous.

We start by identifying $\mathrm{Hom}[\mathcal{EG}, G]$ with $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$, where $\bar{\mathcal{G}}_{x_0}$ is the subgroup of $\bar{\mathcal{G}}$ (17) of the elements g such that $g(x_0) = \mathbf{1}$. Let us fix a unique edge $e_x \in \mathrm{Hom}[x_0, x]$ for each $x \in \Sigma$, e_{x_0} being the trivial hoop. Let us denote this set of edges by $\Lambda = \{e_x, x \in \Sigma\}$. Consider the map

$$\Theta_\Lambda : \mathrm{Hom}[\mathcal{EG}, G] \to \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}, \tag{20}$$

where $\bar{A} \in \mathrm{Hom}[\mathcal{EG}, G]$ is mapped to $(H, g) \in \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ such that

$$H(\beta) = \bar{A}(\beta) \quad \forall \beta \in \mathcal{HG}, \tag{21}$$

and

$$g(x) = \bar{A}(e_x) \quad \forall x \in \Sigma. \tag{22}$$

Consider also the natural action of $\bar{\mathcal{G}}$ on $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ given by

$$(\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}) \times \bar{\mathcal{G}} \ni ((H, g), g') \mapsto (H_{g'}, g_{g'}), \tag{23}$$

where

$$H_{g'}(\beta) = g'(x_0)^{-1} H(\beta) g'(x_0) \quad \forall \beta \in \mathcal{HG}, \tag{24}$$

and

$$g_{g'}(x) = g'(x)^{-1} g(x) g'(x_0) \quad \forall x \in \Sigma. \tag{25}$$

Theorem 1. *For any choice of the set Λ, the map Θ_Λ is a homeomorphism, equivariant with respect to the action of $\bar{\mathcal{G}}$.*

Proof. It is fairly easy to see that Θ_Λ is bijective and equivariant: for a given Λ, the map Θ_Λ is clearly well-defined and its inverse is given by $(H, g) \mapsto \bar{A}$ where

$$\bar{A}(\gamma) = g(r(\gamma)) H(e_{r(\gamma)}^{-1} \gamma e_{s(\gamma)}) g(s(\gamma))^{-1} \quad \forall \gamma \in \mathcal{EG}. \tag{26}$$

It is also clear that Θ_Λ is equivariant with respect to the action of $\bar{\mathcal{G}}$ on $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ ((24) and (25)) and on $\mathrm{Hom}[\mathcal{EG}, G]$ ((18) and (19)). It remains to be shown that Θ_Λ is a homeomorphism. Recall that the topologies of $\mathrm{Hom}[\mathcal{HG}, G]$ and $\mathrm{Hom}[\mathcal{EG}, G]$ are defined by the projective families $\{\chi_S\}_{S\in\mathcal{S}_\mathcal{H}}$ and $\{\mathcal{A}_L\}_{L\in\mathcal{L}}$ considered previously.

Given $S \in \mathcal{S}_\mathcal{H}$ and $x \in \Sigma$, let P_S and π_x, respectively, be the projections from $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ to χ_S and G_x (the copy of G associated with the point x). Recall that the topology of $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ is the weakest topology such that all the maps P_S and π_x are continuous. So, Θ_Λ is continuous if and only if the maps $P_S \circ \Theta_\Lambda$ and $\pi_x \circ \Theta_\Lambda$ are continuous $\forall S \in \mathcal{S}_\mathcal{H}$ and $\forall x \in \Sigma$. Likewise, Θ_Λ^{-1} is continuous if and only if all the maps $p_L \circ \Theta_\Lambda^{-1} : \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \to \mathcal{A}_L$ are continuous, where the projections $p_L : \mathrm{Hom}[\mathcal{EG}, G] \to \mathcal{A}_L$ are defined in (12).

It is straightforward to show that the maps $\pi_x \circ \Theta_\Lambda$ are continuous: given $x \in \Sigma$, one just has to consider the subgroupoid $L = \mathcal{EG}\{e_x\}$ generated by the edge $e_x \in \Lambda$ and the homeomorphism (6) $\rho_{e_x} : \mathcal{A}_L \to G$. It is clear that $\pi_x \circ \Theta_\Lambda$ coincides with $\rho_{e_x} \circ p_L$ being, therefore, continuous.

On the other hand, to show that $P_S \circ \Theta_\Lambda$ and $p_L \circ \Theta_\Lambda^{-1}$ are continuous one needs to consider explicitly the relation between the spaces $\mathcal{A}_L$ and χ_S, $L \in \mathcal{L}$, $S \in \mathcal{S}_\mathcal{H}$.

Lemma 3. *For every $S \in \mathcal{S}_\mathcal{H}$ there exists a connected subgroupoid $L \in \mathcal{L}$ such that S is a subgroup of L. The projection*

$$p_{S,L} : \mathcal{A}_L \to \chi_S \tag{27}$$

defined by the restriction of elements of $\mathcal{A}_L$ to the subgroup S is continuous and satisfies

$$P_S \circ \Theta_\Lambda = p_{S,L} \circ p_L \tag{28}$$

for every Λ.

Proof. Let us consider a set $\{\beta_1, \dots, \beta_n\}$ of independent hoops generating the group S. For each β_i let us fix a piecewise analytic loop ℓ_i in the equivalence class β_i and let σ_i be the corresponding image in Σ. We choose a set $\{e_1, \dots, e_m\}$ of independent edges that decompose $\cup_{i=1}^n \sigma_i$, i.e. $\cup_{i=1}^n \sigma_i = \cup_{j=1}^m \sigma(e_j)$, and denote the connected groupoid $\mathcal{EG}\{e_1, \dots, e_m\} \in \mathcal{L}$ by L. Since the hoops β_i can be obtained as compositions of edges e_j, S is a subgroup of the group $\mathrm{Hom}_L[x_0, x_0]$ of all arrows of L that start and end at x_0. The generators of L define an homeomorphism (6) between $\mathcal{A}_L$ and G^m and the generators of S give us an homeomorphism between χ_S and G^n. The same arguments used to prove the continuity of the maps $p_{L,L'}$ show that the projection $p_{S,L} : \mathcal{A}_L \to \chi_S$ is continuous (see Eq. (10)). Relation (28) is obvious. The independence with respect to Λ follows from the fact that the map $p_{S,L}$ is independent of Λ. □

The continuity of the maps $P_S \circ \Theta_\Lambda$ $\forall S \in \mathcal{S}_\mathcal{H}$, follows immediately from Lemma 3. To show that the maps $p_L \circ \Theta_\Lambda^{-1}$ are continuous one needs the converse of Lemma 3. We will use the following notation. Given a subgroupoid $\Gamma \subset \mathcal{EG}$, $\mathrm{Obj}\,\Gamma$ denotes the set of objects of Γ (the set of all points of Σ which are range or source for some arrow in Γ); $\mathrm{Hom}_\Gamma[x, y]$ stands for the set of all arrows of Γ that start at x and end at y and Π_Γ denotes the natural projection from $\bar{\mathcal{G}}_{x_0}$ to the subgroup $\bar{\mathcal{G}}_{x_0}(\Gamma)$ of all maps $\mathrm{Obj}\,\Gamma \to G$ such that $g(x_0) = \mathbf{1}$. Notice that, as in Theorem 1, given a set $\{\gamma_x, x \in \mathrm{Obj}\,\Gamma\}$ of arrows of Γ, with $\gamma_{x_0} = \mathbf{1}_{x_0}$, one can define a bijection between $\mathrm{Hom}[\Gamma, G]$ and $\mathrm{Hom}[\mathrm{Hom}_\Gamma[x_0, x_0], G] \times \bar{\mathcal{G}}_{x_0}(\Gamma)$ (in this case we use general arrows instead of edges since some of the sets $\mathrm{Hom}_\Gamma[x_0, x]$ may not contain any edges).

Lemma 4. *For every $L \in \mathcal{L}$ there exists $S \in \mathcal{S}_\mathcal{H}$ and a connected subgroupoid $\Gamma \subset \mathcal{EG}$, with $\mathrm{Obj}\,\Gamma = \mathrm{Obj}\,L\dot{\cup}\{x_0\}$, such that $L \subset \Gamma$ and $\mathrm{Hom}_\Gamma[x_0, x_0] = S$. The natural projection from $\mathrm{Hom}[\Gamma, G]$ to $\mathcal{A}_L$ defines a map*

$$p_{L,S} : \chi_S \times \bar{\mathcal{G}}_{x_0}(\Gamma) \to \mathcal{A}_L, \tag{29}$$

which is continuous and satisfies

$$p_L \circ \Theta_\Lambda^{-1} = p_{L,S} \circ (p_S \times \Pi_\Gamma) \tag{30}$$

for an appropriate choice of Λ.

Proof. Let us consider a set $a(L)$ of independent edges generating the groupoid L. If x_0 is an object L, we take $a(L)$ such that no edges in $a(L)$ end at x_0, which is always possible, reverting the orientations of some edges if necessary. Let us consider the subset of $\mathrm{Obj}\,\Gamma$ of the objects that are not connected to x_0 by an edge in $a(L)$. For each such object x, let us add to the set $a(L)$ one edge from x_0 to x, and denote by $\bar{a}(L)$ the set of edges thus obtained. Of course, one can always choose the new edges such that the set $\bar{a}(L)$ remains independent. The image in Σ of the set $\bar{a}(L)$ is thus a connected graph, and x_0 is a vertex of this graph. For each object x of L, $x \neq x_0$, let us choose among the set $\bar{a}(L)$ an unique edge from x_0 to x, and call it e_x. Let e_{x_0} be the trivial hoop and $\Lambda(L) := \{e_x, x \in \mathrm{Obj}\,L\dot{\cup}\{x_0\}\}$. Let $\{e_1, \dots, e_k\}$ be the subset of $a(L)$ of the edges that do not belong to $\Lambda(L)$. With the edges e_i and e_x we construct the hoops

$$\beta_i := e_{r(e_i)}^{-1} e_i e_{s(e_i)}, \quad i = 1, \dots, k. \tag{31}$$

By construction, the set of hoops $\{\beta_1, \dots, \beta_k\}$ is independent. Let S be the subgroup of $\mathcal{HG}$ generated by $\{\beta_1, \dots, \beta_k\}$. From Lemma 2, the set Γ of arrows of the form $e_x \beta e_y^{-1}$, with $\beta \in S$ and $x, y \in \mathrm{Obj}\,L \cup \{x_0\}$, is a connected groupoid such that $\mathrm{Obj}\,\Gamma = \mathrm{Obj}\,L\dot{\cup}\{x_0\}$ and $\mathrm{Hom}_\Gamma[x_0, x_0] = S$. The groupoid L is a subgroupoid of Γ, since all the generators of L belong to Γ, as we show next. For the edges in $a(L)$ that belong also to $\Lambda(L)$ one has $e_x = e_x \mathbf{1}_{x_0} e_{x_0}^{-1} \in \Gamma$. If, on the other hand, the edge is of the type $e_i \in \{e_1, \dots, e_k\}$, then $e_i = e_{r(e_i)} \beta_i e_{s(e_i)}^{-1} \in \Gamma$. We have, therefore, proved that there exist S and Γ such that $L \subset \Gamma$ and $\mathrm{Hom}_\Gamma[x_0, x_0] = S$. For the remaining of the proof, let $p_{L,\Gamma} : \mathrm{Hom}[\Gamma, G] \to \mathcal{A}_L$ be the projection defined by restriction to L and let

$$\Theta_{\Lambda(L)}(\Gamma) : \mathrm{Hom}[\Gamma, G] \to \chi_S \times \bar{\mathcal{G}}_{x_0}(\Gamma) \tag{32}$$

be the bijection associated to the set $\Lambda(L)$. We introduce also the notation

$$p_{L,S} := p_{L,\Gamma} \circ \Theta_{\Lambda(L)}^{-1}(\Gamma) : \chi_S \times \bar{\mathcal{G}}_{x_0}(\Gamma) \to \mathcal{A}_L. \tag{33}$$

Since $\chi_S \times \bar{\mathcal{G}}_{x_0}(\Gamma)$ and $\mathcal{A}_L$ can be identified with powers of G, we conclude, as in the proof of Lemma 3, that $p_{L,S}$ is continuous. Finally, to prove (30) one just has to consider a set of edges Λ that contains $\Lambda(L)$. □

Given that the projections $p_S : \mathrm{Hom}[\mathcal{HG}, G] \to \chi_S$ and $\Pi_\Gamma : \bar{\mathcal{G}}_{x_0} \to \bar{\mathcal{G}}_{x_0}(\Gamma)$ are continuous, Lemma 4 shows that for every fixed $L \in \mathcal{L}$ there exists a Λ such that $p_L \circ \Theta_\Lambda^{-1}$ is continuous, which still does not prove that all the maps $p_L \circ \Theta_\Lambda^{-1}$, $L \in \mathcal{L}$, are continuous for a given Λ. We have, however, the following lemma.

Lemma 5. *The map $\Theta_\Lambda \circ \Theta_{\Lambda'}^{-1}$ is a homeomorphism for any Λ and Λ'.*

Proof. Notice that the map

$$\Theta_\Lambda \circ \Theta_{\Lambda'}^{-1} : \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \to \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \tag{34}$$

is given by

$$(H, g) \mapsto (H', g') \tag{35}$$

such that

$$H' = H, \qquad g'(x) = g(x)H(e_x^{-1}e'_x) \quad \forall x \in \Sigma, \tag{36}$$

where $e_x \in \Lambda$ and $e'_x \in \Lambda'$. It is then sufficient to show that the maps $\pi_x \circ \Theta_\Lambda \circ \Theta_{\Lambda'}^{-1}$ are continuous $\forall x \in \Sigma$, since $P_S \circ \Theta_\Lambda \circ \Theta_{\Lambda'}^{-1} = P_S \, \forall S \in \mathcal{S}_\mathcal{H}$. But $\pi_x \circ \Theta_\Lambda \circ \Theta_{\Lambda'}^{-1}$ can be obtained as composition of the maps

$$\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \ni (H, g) \mapsto (H(e_x^{-1}e'_x), g(x)) \in G \times G, \tag{37}$$

and

$$G \times G \ni (g_1, g_2) \mapsto g_2 g_1 \in G, \tag{38}$$

which are clearly continuous. □

From Lemma 5, we have the following corollary.

Corollary 1. *The continuity of $p_L \circ \Theta_\Lambda^{-1}$ is equivalent to the continuity of $p_L \circ \Theta_{\Lambda'}^{-1}$, for any other values of Λ'.*

This corollary, together with Lemma 4, shows that, for a given Λ, all the maps $p_L \circ \Theta_\Lambda^{-1}$, $L \in \mathcal{L}$, are continuous, which concludes the proof of Theorem 1.

The identification of $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ with $\mathrm{Hom}[\mathcal{HG}, G]/G$ now follows easily. Consider a fixed Λ. Since Θ_Λ is a homeomorphism equivariant with respect to the continuous action of $\bar{\mathcal{G}}$, we conclude that $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is homeomorphic to $(\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0})/\bar{\mathcal{G}}$. On the other hand, it is clear that

$$\frac{\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}}{\bar{\mathcal{G}}} = \frac{\mathrm{Hom}[\mathcal{HG}, G]}{G} \times \frac{\bar{\mathcal{G}}_{x_0}}{\bar{\mathcal{G}}_{x_0}} \cong \frac{\mathrm{Hom}[\mathcal{HG}, G]}{G}. \tag{39}$$

Thus, as a corollary of Theorem 1 one gets the following theorem.

Theorem 2. *The spaces $\mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ and $\mathrm{Hom}[\mathcal{HG}, G]/G$ are homeomorphic.*

It is also interesting to note that the identification $\mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \cong \mathrm{Hom}[\mathcal{EG}, G]$, through the choice of a set of edges $\Lambda = \{e_x, x \in \Sigma\}$ as above, provides a (almost) global gauge-fixing, meaning that there are sections $\eta : \mathrm{Hom}[\mathcal{HG}, G] \to \mathrm{Hom}[\mathcal{EG}, G]$ such that $\mathcal{P} \circ \eta = \mathbf{id}$, where $\mathcal{P} : \mathrm{Hom}[\mathcal{EG}, G] \to \mathrm{Hom}[\mathcal{HG}, G]$ is the canonical projection. $\mathrm{Hom}[\mathcal{HG}, G]$ can, therefore, be identified with a subset of $\mathrm{Hom}[\mathcal{EG}, G]$. In fact, since the edges e_x in the set Λ are algebraically independent, the space $\mathrm{Hom}[\mathcal{HG}, G]$ can be seen as a subset of $\mathrm{Hom}[\mathcal{EG}, G]$ of all generalized connections with given pre-assigned values on the set Λ. Choosing, for instance, the identity of G for all e_x, one then has the identification

$$\mathrm{Hom}[\mathcal{HG}, G] \cong \{\bar{A} \in \mathrm{Hom}[\mathcal{EG}, G] | \bar{A}(e_x) = \mathbf{1} \quad \forall x \in \Sigma\}. \tag{40}$$

There remains, of course, the non-trivial action of gauge transformations at the base point x_0. A study of the action of the full gauge group $\bar{\mathcal{G}}$ was recently done by Fleischhack [11,13], leading to stratification results in the context of generalized connections (see also [20]). A detailed account on the existence of Gribov ambiguities when the full gauge-invariant space $\mathrm{Hom}[\mathcal{HG}, G]/G \cong \mathrm{Hom}[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is considered is given in [14].

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